

# On Sylvester Colorings of Cubic Graphs

Anush Hakobyan\*, Vahan Mkrtchyan

*Department of Informatics and Applied Mathematics, Yerevan State University, Yerevan, 0025, Armenia*

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## Abstract

If  $G$  and  $H$  are two cubic graphs, then an  $H$ -coloring of  $G$  is a proper edge-coloring  $f$  with edges of  $H$ , such that for each vertex  $x$  of  $G$ , there is a vertex  $y$  of  $H$  with  $f(\partial_G(x)) = \partial_H(y)$ . If  $G$  admits an  $H$ -coloring, then we will write  $H \prec G$ . The Petersen coloring conjecture of Jaeger states that for any bridgeless cubic graph  $G$ , one has:  $P \prec G$ . The second author has recently introduced the Sylvester coloring conjecture, which states that for any cubic graph  $G$  one has:  $S \prec G$ . Here  $S$  is the Sylvester graph on ten vertices. In this paper, we prove the analogue of Sylvester coloring conjecture for cubic pseudo-graphs. Moreover, we show that if  $G$  is any connected simple cubic graph  $G$  with  $G \prec P$ , then  $G = P$ . This implies that the Petersen graph does not admit an  $S_{16}$ -coloring, where  $S_{16}$  is the Sylvester graph on 16 vertices. Finally, we show that any cubic graph  $G$  has a coloring with edges of Sylvester graph  $S$  such that at least  $\frac{4}{5}$  of vertices of  $G$  meet the conditions of Sylvester coloring conjecture.

*Keywords:* Cubic graph; Petersen graph; Petersen coloring conjecture; Sylvester graph; Sylvester coloring conjecture

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## 1. Introduction

Graphs considered in this paper are finite and undirected. They do not contain loops, though they may contain parallel edges. We also consider pseudo-graphs, which may contain both loops and parallel edges, and simple graphs, which contain neither loops nor parallel edges. As usual, a loop contributes to the degree of a vertex by 2.

Within the frames of this paper, we assume that graphs, pseudo-graphs and simple graphs are considered up to isomorphisms. This implies that the equality  $G = G'$  means that  $G$  and  $G'$  are isomorphic.

For a graph  $G$ , let  $V(G)$  and  $E(G)$  be the set of vertices and edges of  $G$ , respectively. Moreover, let  $\partial_G(x)$  be the set of edges of  $G$  that are incident to the vertex  $x$  of  $G$ . A

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<sup>\*</sup>Corresponding author

*Email addresses:* ashunik94@gmail.com (Anush Hakobyan), vahanmkrtchyan2002@ysu.am (Vahan Mkrtchyan)

matching of  $G$  is a set of edges of  $G$  such that any two of them do not share a vertex. A matching of  $G$  is perfect, if it contains  $\frac{|V(G)|}{2}$  edges. For a positive integer  $k$ , a  $k$ -factor of  $G$  is a spanning  $k$ -regular subgraph of  $G$ . Observe that the edge-set of a 1-factor of  $G$  is a perfect matching of  $G$ . Moreover, if  $G$  is cubic and  $F$  is a 1-factor of  $G$ , then the set  $E(G) \setminus E(F)$  is an edge-set of a 2-factor of  $G$ . This 2-factor is said to be complementary to  $F$ . Conversely, if  $\bar{F}$  is a 2-factor of a cubic graph  $G$ , then the set  $E(G) \setminus E(\bar{F})$  is an edge-set of a 1-factor of  $G$  or is a perfect matching of  $G$ . This 1-factor is said to be complementary to  $\bar{F}$ .

If  $P$  is a path of a graph  $G$ , then the length of  $P$  is the number of edges of  $G$  lying on  $P$ . For a connected graph  $G$  and its two vertices  $u$  and  $v$ , the distance between  $u$  and  $v$  is the length of the shortest path connecting these vertices. The distance between edges  $e$  and  $f$  of  $G$ , denoted by  $\rho_G(e, f)$ , is the shortest distance among end-vertices of  $e$  and  $f$ . Clearly, adjacent edges are at distance zero.

A subgraph  $H$  of  $G$  is even, if every vertex of  $H$  has even degree in  $H$ . A block of  $G$  is a maximal 2-connected subgraph of  $G$ . An end-block is a block of  $G$  containing at most one vertex that is a cut-vertex of  $G$ . If  $G$  is a cubic graph containing cut-vertices, then any end-block  $B$  of  $G$  is adjacent to a unique bridge  $e$ . We will refer to  $e$  as a bridge corresponding to  $B$ . Moreover, if  $e = (u, v)$  and  $u \in V(B)$ ,  $v \notin V(B)$ , then  $v$  is called the root of  $B$ .

If  $G$  is a cubic graph, and  $K$  is a triangle in  $G$ , then one can obtain a cubic pseudo-graph by contracting  $K$ . We will denote this pseudo-graph by  $G/K$ . If  $G/K$  is a graph, we will say that  $K$  is contractible. Observe that if  $K$  is not contractible, two vertices of  $K$  are joined with two parallel edges, and the third vertex is incident to a bridge (see the end-blocks of the graph from Figure 2). If  $K$  is a contractible triangle, and  $e$  is an edge of  $K$ , then let  $f$  be the edge of  $G$  that is incident to a vertex of  $K$  and is not adjacent to  $e$ .  $e$  and  $f$  will be called opposite edges.

If  $T$  is a set,  $H$  is a subgraph of a graph  $G$ , and  $f : E(G) \rightarrow T$ , then a mapping  $g : E(H) \rightarrow T$ , such that  $g(e) = f(e)$  for any  $e \in E(H)$  is called the restriction of  $f$  to  $H$ .

Let  $G$  and  $H$  be two cubic graphs, and let  $f : E(G) \rightarrow E(H)$ . Define:

$$V(f) = \{x \in V(G) : \exists y \in V(H) f(\partial_G(x)) = \partial_H(y)\}.$$

An  $H$ -coloring of  $G$  is a mapping  $f : E(G) \rightarrow E(H)$ , such that  $V(f) = V(G)$ . If  $G$  admits an  $H$ -coloring, then we will write  $H \prec G$ .

If  $H \prec G$  and  $f$  is an  $H$ -coloring of  $G$ , then for any adjacent edges  $e, e'$  of  $G$ , the edges  $f(e), f(e')$  of  $H$  are adjacent. Moreover, if the graph  $H$  contains no triangle, then the converse is also true, that is, if a mapping  $f : E(G) \rightarrow E(H)$  has a property that for any two adjacent edges  $e$  and  $e'$  of  $G$ , the edges  $f(e)$  and  $f(e')$  of  $H$  are adjacent, then  $f$  is an  $H$ -coloring of  $G$ .

Let  $P$  be the well-known Petersen graph (Figure 1) and let  $S$  be the graph from Figure 2.  $S$  is called the Sylvester graph [7]. We would like to point out that usually the name Sylvester graph is used for a particular strongly regular graph on 36 vertices, and this graph should not be confused with  $S$ , which has 10 vertices.

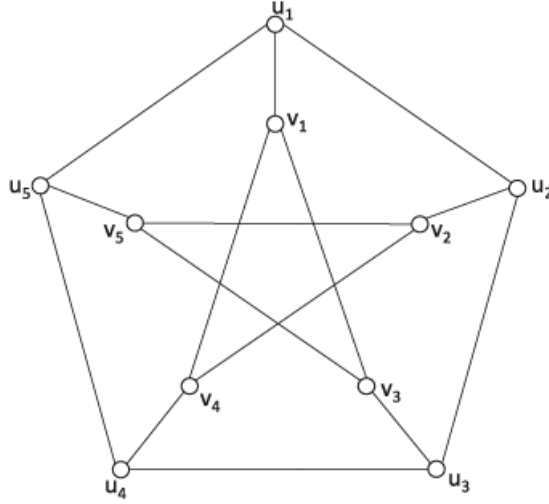


Figure 1: The Petersen Graph

The Petersen coloring conjecture of Jaeger states:

**Conjecture 1.** (Jaeger, 1988 [4]) *For each bridgeless cubic graph  $G$ , one has  $P \prec G$ .*

The conjecture is difficult to prove, since it can be seen that it implies the following two classical conjectures:

**Conjecture 2.** (Berge-Fulkerson, 1972 [2, 9]) *Any bridgeless cubic graph  $G$  contains six (not necessarily distinct) perfect matchings  $F_1, \dots, F_6$  such that any edge of  $G$  belongs to exactly two of them.*

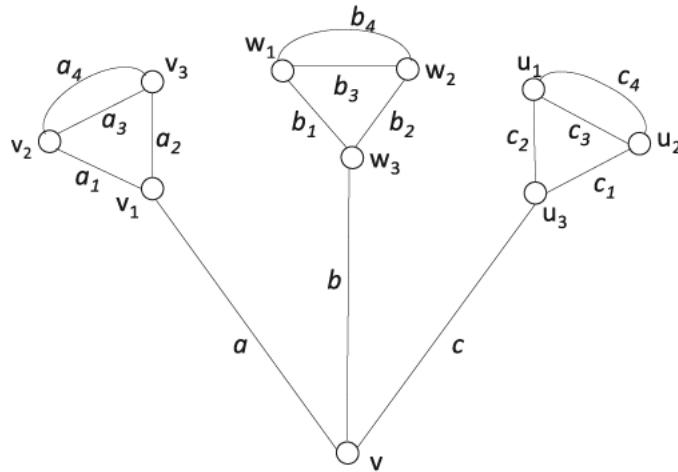


Figure 2: The Sylvester Graph

**Conjecture 3.** (*(5, 2)-cycle-cover conjecture, [8, 12]*) Any bridgeless graph  $G$  (not necessarily cubic) contains five even subgraphs such that any edge of  $G$  belongs to exactly two of them.

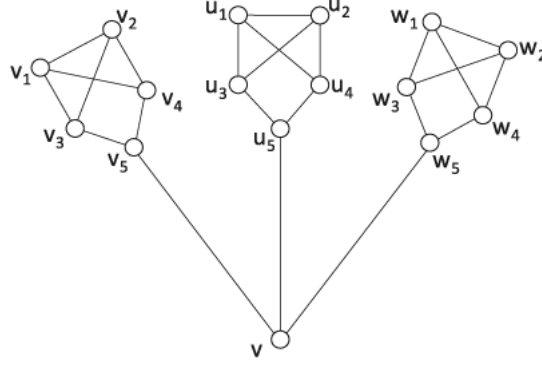


Figure 3: The Sylvester Graph on 16 Vertices

Related with the Jaeger conjecture, the following conjecture has been introduced in [6]:

**Conjecture 4.** (*V. V. Mkrtchyan, 2012 [6]*) For each cubic graph  $G$ , one has  $S \prec G$ .

In direct analogy with the Jaeger conjecture, we call conjecture 4 the Sylvester coloring conjecture.

In this paper, we consider the analogues of this conjecture for simple cubic graphs and cubic pseudo-graphs. Let  $S_{16}$  be the Sylvester's simple graph on 16 vertices (Figure 3) and let  $S_4$  be the Sylvester's pseudo-graph on 4 vertices (Figure 4).

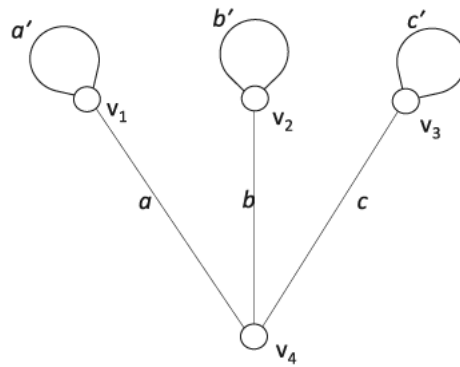


Figure 4: Sylvester's Pseudo Graph on 4 Vertices

In this paper, we show that not all simple cubic graphs admit an  $S_{16}$ -coloring. On the positive side, we prove that all cubic pseudo-graphs have an  $S_4$ -coloring. In the last statement, the term  $S_4$ -coloring should be understood as follows: loops receive two colors, and ordinary edges receive one color. Moreover, if a loop of  $S_4$  is to be used in an  $S_4$ -coloring of  $G$ , then in the neighborhood of the corresponding vertex of  $G$  it is used twice. We complete the paper by proving a result towards conjecture 4. It states that for any cubic graph  $G$  there is a mapping  $f : E(G) \rightarrow E(S)$ , such that  $|V(f)| \geq \frac{4}{5} \cdot |V(G)|$ .

Terms and concepts that we do not define in the paper can be found in [3, 13].

## 2. Some Auxiliary Statements

In this section, we present some auxiliary statements that will be used in Section 3.

**Theorem 1.** (*Petersen, 1891 [5]*) *Let  $G$  be a cubic graph containing at most two bridges. Then  $G$  has a 1-factor.*

**Lemma 1.** *Let  $G$  be a bridgeless graph with  $d(v) \in \{2, 3\}$  for any  $v \in V(G)$ . Assume that all vertices of  $G$  are of degree 3 except one. Then  $G$  has a 2-factor.*

**Proof:** Take two copies  $G_1$  and  $G_2$  of  $G$ , and consider a graph  $H$  obtained from them by joining degree 2 vertices by an edge  $e$ . Observe that  $H$  is a cubic graph containing only one bridge, which is the edge  $e$ . By Theorem 1,  $H$  contains a 1-factor  $F$ . Since  $e$  is a bridge of  $H$ , we have  $e \in F$ . Consider the complementary 2-factor  $\bar{F}$  of  $F$ . Clearly, the edges of the set  $E(\bar{F}) \cap E(G_1)$  form a 2-factor of  $G_1$ , which completes the proof of the lemma.  $\square$

**Proposition 1.** *Let  $G$  be a simple cubic graph that has no a perfect matching and  $|V(G)| \leq 16$ . Then  $G = S_{16}$ .*

**Lemma 2.** *Suppose that  $G$  and  $H$  are cubic graphs with  $H \prec G$ , and let  $f$  be an  $H$ -coloring of  $G$ . Then:*

- (a) *If  $M$  is any matching of  $H$ , then  $f^{-1}(M)$  is a matching of  $G$ ;*
- (b)  *$\chi'(G) \leq \chi'(H)$ , where  $\chi'(G)$  is the chromatic index of  $G$ ;*
- (c) *If  $M$  is a perfect matching of  $H$ , then  $f^{-1}(M)$  is a perfect matching of  $G$ ;*
- (d) *For every even subgraph  $C$  of  $H$ ,  $f^{-1}(C)$  is an even subgraph of  $G$ ;*
- (e) *For every bridge  $e$  of  $G$ , the edge  $f(e)$  is a bridge of  $H$ .*

**Proposition 2.** *Let  $G$  be a connected non-3-edge-colorable simple cubic graph such that  $|V(G)| \leq 10$ . Then  $G = P$  or  $G = S'$  (Figure 5).*

We will also need some results that were obtained in [1, 10, 11]. Let  $G$  be a graph of maximum degree at most 3, and assume that  $c$  is a coloring of some edges of  $G$  with colors 1, 2 and 3. The edges of  $G$  that have not received a color in  $c$  are called uncolored edges. Now, assume that  $c$  is chosen so that the number of uncolored edges of  $G$  is minimized. It is known that for such a choice of  $c$ , uncolored edges must form a matching.

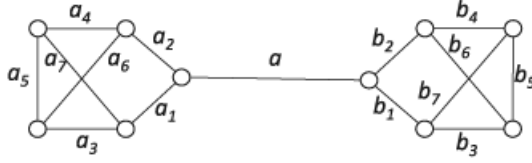


Figure 5: The graph  $S'$  on 10 vertices

Take an arbitrary uncolored edge  $e = (u, v)$ . As  $c$  is chosen so that the number of uncolored is smallest, we have that  $e$  is incident to edges that have colors 1, 2, and 3. Since  $G$  is of maximum degree at most 3, we have that there are colors  $\alpha, \beta \in \{1, 2, 3\}$ , such that no edge incident to  $u$  and  $v$  is colored with  $\beta$  and  $\alpha$ , respectively. Consider a maximal  $\alpha - \beta$ -alternating path  $P_e$  starting from  $u$ . As it shown in [1, 10, 11], this path must terminate in  $v$ , and hence it is of even length. This means that  $P_e$  together with the edge  $e$  forms an odd cycle  $C_e$ .  $C_e$  is called the cycle corresponding to the uncolored edge  $e$ . It is known that

**Lemma 3.** ([1, 10, 11]) *Cycles corresponding to different uncolored edges are vertex-disjoint.*

### 3. The Main Results

In this section, we obtain the main results of the paper. Our first theorem shows that the statement analogous to Sylvester coloring conjecture holds for cubic pseudo-graphs.

**Theorem 2.** *Let  $G$  be a cubic pseudo-graph. Then  $G$  admits an  $S_4$ -coloring.*

**Proof:** It is clear that we can prove the theorem only for connected cubic pseudo-graphs  $G$ . We show that without loss of generality, we can assume that  $G$  is a graph (that is,  $G$  does not contain loops). If  $G$  contains loops, then consider a graph  $H$  obtained from  $G$  by replacing all vertices of  $G$  incident to loops by triangles (Figure 6).

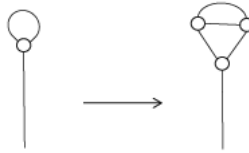


Figure 6: Modification of loops of  $G$

Now  $H$  has an  $S_4$ -coloring by assumption. Consider a coloring of edges of  $G$  by coloring all non-loop edges with the same color as they had in  $H$ , and by coloring the loops with the two colors of edges that are incident to the corresponding bridge of  $H$  (Figure 6). It is not hard to see that the described coloring is an  $S_4$ -coloring of  $G$ .

Thus we can assume that  $G$  is a graph. In order to complete the proof of the theorem, we proceed by induction on the number of bridges of  $G$ . We will use the labels of edges of  $S_4$  given in Figure 4.

If there are at most two bridges in  $G$ , then due to Theorem 1, the graph  $G$  has a 1-factor. Color the edges of the 1-factor by  $a$ , and the edges of the complementary 2-factor by  $a'$ . It is not hard to see that the described coloring is an  $S_4$ -coloring of  $G$ .

Now assume that the statement is true for graphs with at most  $k$  bridges, and we prove it for those with  $k + 1$  bridges ( $k + 1 \geq 3$ ). We will consider the following cases:

*Case 1:* For any two end-blocks  $B$  and  $B'$  of  $G$ , the bridges  $e$  and  $e'$  corresponding to them, are adjacent.

It is not hard to see that in this case,  $G$  consists of three end-blocks  $B_1, B_2, B_3$ , such that the bridges  $e_1, e_2, e_3$  corresponding to them are incident to the same cut-vertex  $v$ . We obtain an  $S_4$ -coloring of  $G$  as follows: let  $\bar{F}_1, \bar{F}_2, \bar{F}_3$  be 2-factors in  $B_1, B_2, B_3$ , respectively (see Lemma 1). Color the edges of  $\bar{F}_1$  with  $a'$  and the edges of  $(E(B_1) \setminus \bar{F}_1) \cup \{e_1\}$  with  $a$ , the edges of  $\bar{F}_2$  with  $b'$  and the edges of  $(E(B_2) \setminus \bar{F}_2) \cup \{e_2\}$  with  $b$ , the edges of  $\bar{F}_3$  with  $c'$  and the edges of  $(E(B_3) \setminus \bar{F}_3) \cup \{e_3\}$  with  $c$ . It is not hard to see that the described coloring is an  $S_4$ -coloring of  $G$ .

*Case 2:* There are two end-blocks  $B$  and  $B'$  of  $G$ , such that the bridges  $e$  and  $e'$  corresponding to them, are not adjacent.

Assume that  $e = (u, u')$ ,  $e' = (v, v')$  and  $u' \in V(B)$ ,  $v' \in V(B')$ . Consider a cubic graph  $H$  obtained from  $G$  as follows:

$$H = [G \setminus (V(B_1) \cup V(B_2))] \cup \{(u, v)\}.$$

If initially  $G$  had an edge  $(u, v)$ , then in  $H$  we will just get two parallel edges between  $u$  and  $v$ .

Observe that  $H$  is a cubic graph containing at most  $k$  bridges. By induction hypothesis,  $H$  admits an  $S_4$ -coloring  $g$ . Now we obtain an  $S_4$ -coloring for the graph  $G$  using the coloring  $g$  of  $H$ . For that purpose we consider 2 cases.

*Subcase 2.1:*  $g((u, v)) \in \{a, b, c\}$ .

Without loss of generality, we can assume that  $g((u, v)) = a$ . Other cases can be come up in a similar way. By Lemma 1,  $B$  and  $B'$  contain 2-factors  $\bar{F}$  and  $\bar{F}'$ , respectively. Consider the restriction of  $g$  to  $G$ . We extend it to an  $S_4$ -coloring of  $G$  as follows: color the edges of  $\bar{F} \cup \bar{F}'$  with  $a'$  and the edges of  $(E(B) \setminus \bar{F}) \cup (E(B') \setminus \bar{F}') \cup \{e, e'\}$  with  $a$ . It is not hard to see that the described coloring is an  $S_4$ -coloring of  $G$ .

*Subcase 2.2:*  $g((u, v)) \in \{a', b', c'\}$ .

Without loss of generality, we can assume that  $g((u, v)) = a'$ . Other cases can be come up in a similar way. It is not hard to see that the edges of  $H$  colored with  $a'$  (the edges of set  $f^{-1}(a')$ ) form vertex disjoint cycles in  $H$ . Consider the cycle  $C_{uv}$  of  $G$  containing the edge  $(u, v)$ , and let  $P_{uv} = C_{uv} - (u, v)$  (Figure 7). By Lemma 1,  $B$  and  $B'$  contain 2-factors  $\bar{F}$  and  $\bar{F}'$ , respectively.

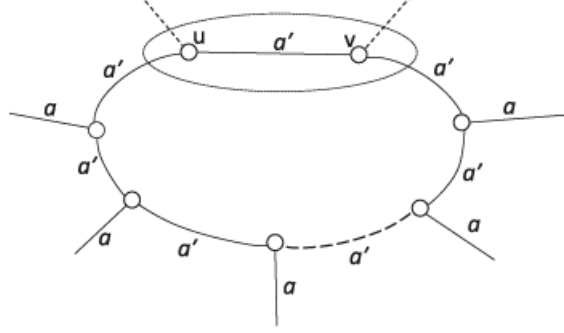


Figure 7: The cycle  $C_{uv}$  and the path  $P_{uv}$  in the graph  $H$

Define edges  $d$  and  $d'$  of  $S_4$  as follows: if  $P_{uv}$  is of odd length, then  $d = b$ ,  $d' = b'$ , and  $d = c$ ,  $d' = c'$ , otherwise. Consider the restriction of  $g$  to  $G$ . We extend it to an  $S_4$ -coloring of  $G$  as follows: color the edges of  $\bar{F}$  with  $b'$  and the edges of  $E(B) \cup \{e\}$  with  $b$ , re-color the edges of  $P_{uv}$  by coloring them with colors  $b$  and  $c$  alternatively beginning from  $c$ , color the edges of  $\bar{F}'$  with  $d'$  and the edges of  $E(B') \cup \{e'\}$  with  $d$  (Figure 8). It is not hard to see that the described coloring is an  $S_4$ -coloring of  $G$ .

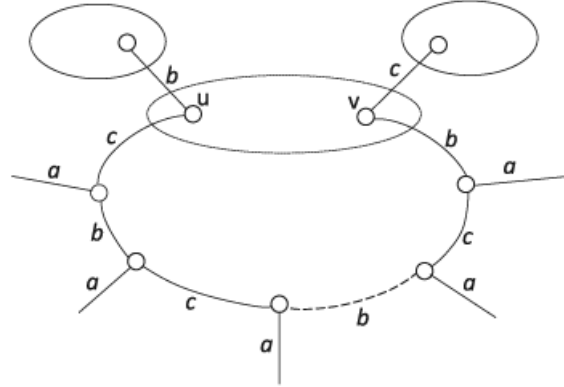


Figure 8: The path  $P_{uv}$  in the graph  $G$

The proof of the theorem is completed.  $\square$

Conjecture 4 states that all cubic graphs admit an  $S$ -coloring. On the other hand, in the previous theorem we have shown that all cubic pseudo-graphs have an  $S_4$ -coloring. One may wonder whether there is a statement analogous to these in the class of simple cubic graphs? More precisely, is there a connected simple cubic graph  $H$  such that all simple cubic graphs admit an  $H$ -coloring? A natural candidate for  $H$  is the graph  $S_{16}$ . Next we prove a theorem that justifies our choice of  $S_{16}$ . On an intuitive level it states that the only way of coloring the graph  $S_{16}$  with some connected simple cubic graph  $H$  is to take  $H = S_{16}$ . This result is analogous to the following theorem proved in [6].



**Theorem 3.** (V. V. Mkrtchyan, [6]) Let  $G$  be a connected cubic graph with  $G \prec S$ , then  $G = S$ .

This is the precise formulation of our second result.

**Theorem 4.** Let  $G$  be a connected simple cubic graph with  $G \prec S_{16}$ , then  $G = S_{16}$ .

**Proof:** As  $G \prec S_{16}$  and  $S_{16}$  has no a perfect matching, then due to (c) of Lemma 2, the graph  $G$  also has no a perfect matching.

Let  $f$  be a  $G$ -coloring of  $S_{16}$ . If  $e \in E(G)$ , then we will say that  $e$  is used (with respect to  $f$ ), if  $f^{-1}(e) \neq \emptyset$ . First of all, let us show that if an edge  $e$  of  $G$  is used, then any edge adjacent to  $e$  is also used.

So let  $e = (u, v)$  be a used edge of  $G$ . For the sake of contradiction, assume that  $v$  is incident to an edge  $z \in E(G)$  that is not used. Assume that  $\partial_G(u) = \{a, b, e\}$  (Figure 9).

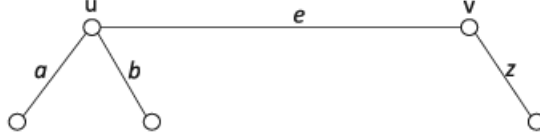


Figure 9: The edge  $e$  in the graph  $G$

We will use the labels of edges of  $S_{16}$  from Figure 3. The following cases are possible:

*Case 1:* The edge  $e$  colors an edge that is not a bridge in  $S_{16}$ . So it is used in end-blocks of  $S_{16}$ . Due to symmetry of  $S_{16}$ , there will be following subcases:

*Subcase 1.1:*  $f((v_1, v_2)) = e$ .

Since  $z$  is not a used edge, we can assume that  $f((v_1, v_3)) = a$  and  $f((v_1, v_4)) = b$ . Since adjacent edges receive different colors and  $z$  is not a used edge, we have that  $f((v_2, v_3)) = b$  and  $f((v_2, v_4)) = a$ . This implies that

$$f((v_3, v_5)) = f((v_4, v_5)) = e,$$

which contradicts the fact that adjacent edges receive different colors.

*Subcase 1.2:*  $f((v_1, v_3)) = e$ .

Since  $z$  is not a used edge, we can assume that  $f((v_1, v_2)) = a$  and  $f((v_1, v_4)) = b$ . As adjacent edges receive different colors and  $z$  is not a used edge, we have that  $f((v_2, v_3)) = b$  and  $f((v_3, v_5)) = a$ . But then  $f((v_2, v_4)) = e$ , which implies that  $f((v_4, v_5)) = a$ . This contradicts the fact that adjacent edges receive different colors.

*Subcase 1.3:*  $f((v_3, v_5)) = e$ .

Since  $z$  is not a used edge, we can assume that  $f((v_1, v_3)) = a$  and  $f((v_2, v_3)) = b$ . Consider the edge  $(v_1, v_2)$ . Observe that its color can be either  $e$ , or there is an edge  $h$  of  $G$ ,

such that  $a$ ,  $b$  and  $h$  form a triangle in  $G$  and  $h$  is not incident to  $u$ . Since we have ruled out Case 1.1, we can assume that the color of  $(v_1, v_2)$  is not  $e$ , hence there is the above-mentioned edge  $h$ . Let  $x$  and  $y$  be the edge of  $G$  that are adjacent to  $b$  and  $h$ , and  $a$  and  $h$ , respectively, that are not incident to  $u$ . Observe that  $f((v_1, v_4)) = y$  and  $f((v_2, v_4)) = x$ . On the other hand, it is not hard to see that since  $z$  is not a used edge and  $f((v_3, v_5)) = e$ , we have that  $f((v_4, v_5)) \in \{a, b\}$ . This is a contradiction since there is no vertex  $w$  of  $G$  such that  $\partial_G(w) = \{a, x, y\}$  or  $\partial_G(w) = \{b, x, y\}$ .

*Case 2:* The edge  $e$  colors an edge that is a bridge in  $S_{16}$ .

The consideration of Case 1 implies that, without loss of generality, we can assume that the edge  $e$  is not used in end-blocks of  $S_{16}$ . Assume that  $f((v, v_5)) = e$ . Since  $z$  is not a used edge, we can assume that  $f((v_3, v_5)) = a$  and  $f((v_4, v_5)) = b$ .

Assume that  $a = (u, u_a)$  and  $b = (u, u_b)$ . We claim that  $u_a$  and  $u_b$  are not joined with an edge in  $G$ . Assume the opposite. Let  $h = (u_a, u_b) \in E(G)$ . Let  $x$  and  $y$  be the edges of  $G$  incident to  $u_a$  and  $u_b$ , respectively, that are different from  $a$  and  $h$ , and  $b$  and  $h$ , respectively. Then, we can assume that

$$f((v_1, v_3)) = x, f((v_2, v_3)) = h,$$

and

$$f((v_2, v_4)) = y, f((v_1, v_4)) = h.$$

This implies that

$$a = f((v_1, v_2)) = b.$$

Hence  $a$  and  $b$  are parallel edges of  $G$ , which contradicts the simpleness of  $G$ . Hence  $u_a$  and  $u_b$  are not joined with an edge in  $G$ . Let  $x$  and  $y$  be the edges of  $G$  incident to  $u_a$ , that are different from  $a$ . Similarly, let  $z$  and  $\alpha$  be the edges of  $G$  incident to  $u_b$ , that are different from  $b$ . We can assume that

$$f((v_1, v_3)) = x, f((v_2, v_3)) = y,$$

and

$$f((v_2, v_4)) = z, f((v_1, v_4)) = \alpha.$$

This implies that  $x$  and  $\alpha$  are sharing a vertex  $u_{x,\alpha}$  of  $G$ ,  $y$  and  $z$  are sharing a vertex  $u_{y,z}$  of  $G$ , and  $u_{x,\alpha}$ ,  $u_{y,z}$  are joined with an edge  $g$  of  $G$ , such that  $f((v_1, v_2)) = g$ . Observe that the edges  $a$  and  $b$  are lying on a cycle of  $G$ . Now, since  $z$  is not a used edge, we have that the other two bridges of  $G$  ( $\neq (v, v_5)$ ) are colored with  $a$  and  $b$ . This contradicts (e) of Lemma 2, since  $a$  and  $b$  are not bridges of  $G$ .

The consideration of above two cases implies that any used edges of  $G$  is adjacent to a used edge. Since  $G$  is connected, we have that all edges of  $G$  are used. Since  $|E(S_{16})| = 24$ , we have that  $|E(G)| \leq 24$ , or  $|V(G)| \leq 16$ . Proposition 1 implies that  $G = S_{16}$ . The proof of the theorem is completed.  $\square$

In [6], the following result is obtained:

**Theorem 5.** (V. V. Mkrtchyan, [6]) *Let  $G$  be a connected bridgeless cubic graph with  $G \prec P$ , then  $G = P$ .*

Below we prove the analogue of this result for simple cubic graphs that may contain bridges. Our strategy of the proof is similar to that of given in [6].

**Theorem 6.** *If  $G$  is a connected simple cubic graph with  $G \prec P$ , then  $G = P$ .*

**Proof:** By (b) of Lemma 2,  $G$  is non-3-edge-colorable. Let  $f$  be a  $G$ -coloring of  $P$ . As in the proof of the previous theorem, we say that an edge  $e \in E(G)$  is used (with respect to  $f$ ) if  $f^{-1}(e) \neq \emptyset$ . First of all, let us show that if an edge of  $G$  is used, then all edges adjacent to it are used.

Suppose that  $e = (u, v)$  is a used edge, and for the sake of contradiction, assume that the edge  $z$  incident to  $v$  is not used. Assume that  $\partial_G(u) = \{a, b, e\}$ . We will make use of labels of vertices of  $P$  given on Figure 1.

Since  $e$  is a used edge, due to symmetry of  $P$ , we can assume that  $f((v_3, v_4)) = e$ ,  $f((v_4, v_5)) = a$  and  $f((u_4, v_4)) = b$ . Since  $z$  is not a used edge, due to symmetry  $P$ , we can assume that  $f((u_3, v_3)) = b$ ,  $f((v_2, v_3)) = a$ .

Define

$$a_1 = f((v_1, v_5)), \text{ and } a_2 = f((v_1, v_2)).$$

Observe that since  $f$  is a  $G$ -coloring of  $P$ , we have that  $a_1$  and  $a_2$  are adjacent edges of  $G$ . Moreover, each of them is adjacent to  $a$ .

Similarly, define the edges

$$b_1 = f((u_1, u_4)), \text{ and } b_2 = f((u_1, u_3)).$$

Again, we have that  $b_1$  and  $b_2$  are adjacent edges of  $G$ . Moreover, each of them is adjacent to  $b$ .

We will consider three cases:

*Case 1:* The edges  $a_1$ ,  $a_2$  and  $a$  do not form a triangle in  $G$ .

Observe that in this case  $f((u_1, v_1)) = a$ . This implies that the edges  $a$ ,  $b_1$ ,  $b_2$  must be incident to the same vertex. However, this is possible only when  $b_1$  and  $b_2$  are two parallel edges. This is a contradiction, since  $G$  is a simple graph.

*Case 2:* The edges  $b_1$ ,  $b_2$  and  $b$  do not form a triangle in  $G$ .

This case is similar to Case 1.

*Case 3:* The edges  $a_1$ ,  $a_2$  and  $a$  form a triangle in  $G$ . Similarly,  $b_1$ ,  $b_2$  and  $b$  form a triangle.

Let  $a_3$  be the edge of  $G$  that is adjacent to  $a_1$ ,  $a_2$  and is not adjacent to  $a$ . Similarly, let  $b_3$  be the edge of  $G$  that is adjacent to  $b_1$ ,  $b_2$  and is not adjacent to  $b$ . Note that the edges  $a_3$  and  $b_3$  exist, since  $G$  is simple.

Observe that

$$a_3 = f((u_1, v_1)) = b_3,$$

hence  $a_3 = b_3$ . Depending on whether  $a$  and  $b$  belong to the same triangle or not, we will consider the following sub-cases:

*Subcase 3.1:*  $a$  and  $b$  belong to different triangles.

Observe that in this case the edge  $e$  must belong to both of them, hence we have the situation depicted on Figure 10. It is not hard to see that in this case  $a_3 \neq b_3$ , which is a contradiction.

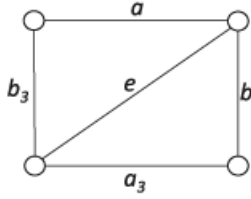


Figure 10:  $a$  and  $b$  belong to different triangles.

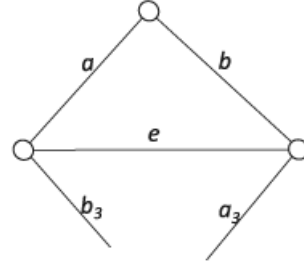


Figure 11:  $a$  and  $b$  belong to the same triangle.

*Subcase 3.2:* Let  $a$  and  $b$  belong to the same triangle (See Figure 11).

In this case  $b_3$  should be adjacent to  $a$ , and  $a_3$  should be adjacent to  $b$ . It is not hard to see that in this case  $a_3 \neq b_3$ , which is a contradiction.

The consideration of above three cases implies that any used edges of  $G$  is adjacent to a used edge. Since  $G$  is connected, we have that all edges of  $G$  are used. Since  $|E(P)| = 15$ , we have that  $|E(G)| \leq 15$ , or  $|V(G)| \leq 10$ . Proposition 2 implies that  $G = P$  or  $G = S'$ .

In order to complete the proof of the theorem, it suffices to show that  $P$  does not admit an  $S'$ -coloring, such that all edges of  $S'$  are used. We will use the labels of edges of  $S'$  given on Figure 5.

For the sake of the contradiction, assume that  $P$  admits an  $S'$ -coloring  $f$ , such that all edges of  $S'$  are used. Due to symmetry of  $P$ , we can assume that  $f((v_3, v_4)) = a$ . Consider the connected components  $A$  and  $B$  of  $S' - a$ . Since  $P$  is 2-edge-connected, there is a vertex  $w$  of  $P$  such that  $w$  is incident to at least one edge that has color from  $A$ , and at least one edge that has color from  $B$ . Observe that this vertex violates the definition of  $S'$ -coloring. This is a contradiction, hence  $P$  does not admit an  $S'$ -coloring as well. The proof of the theorem is completed.  $\square$

The theorem proved above implies that

**Corollary 1.**  *$P$  does not admit an  $S_{16}$ -coloring.*

This corollary and Theorem 4 suggest that a statement analogous to Sylvester coloring conjecture is impossible in the class of simple cubic graphs.

In the end of the paper, we obtain our final result. It states that any cubic graph admits a coloring with edges of  $S$ , such that 80% of vertices meet the constraints of Sylvester coloring conjecture.

**Theorem 7.** *Let  $G$  be a cubic graph. Then, there is a mapping  $f : E(G) \rightarrow S$ , such that*

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|,$$

*and for any  $v \in V(G) \setminus V(f)$  there are two edges  $e, e' \in \partial_G(v)$ , such that  $f(e) = f(e')$ .*

**Proof:** We prove the theorem by induction on the number of vertices. If  $|V(G)| = 2$ , we can take an arbitrary vertex  $w$  of  $S$ , and color the three edges of  $G$  with edges incident to  $w$ . It is trivial to see that this coloring satisfies the condition of the theorem. Now, assume that the statement of the theorem holds for all cubic graphs with  $|V(G)| < n$ , and consider an arbitrary cubic graph  $G$  containing  $n \geq 4$  vertices.

We will consider two cases.

*Case 1:*  $G$  contains a contractible triangle  $T$ .

Consider the cubic graph  $H = G/T$ , and let  $v_T$  be the vertex of  $H$  obtained by contracting  $T$ . Since  $H$  contains  $n - 2$  vertices, we have that there is a mapping  $g : E(H) \rightarrow S$ , such that

$$|V(g)| \geq \frac{4}{5} \cdot |V(H)|,$$

and for any  $v \in V(H) \setminus V(g)$  there are two edges  $e, e' \in \partial_H(v)$ , such that  $g(e) = g(e')$ . We will consider 2 subcases.

*Subcase 1.1:*  $v_T \in V(g)$ .

There is a vertex  $s \in V(S)$ , such that  $g(\partial_H(v_T)) = \partial_S(s)$ . Let  $\partial_S(s) = \{\alpha, \beta, \gamma\}$ . Consider a mapping  $f : E(G) \rightarrow S$ , obtained from  $g$  as follows: color the edges of  $T$  with a color from  $\{\alpha, \beta, \gamma\}$ , such that its end-vertices are not incident to an edge with that color. Observe that

$$|V(f)| = |V(g)| + 2, \text{ and } |V(G)| = |V(H)| + 2,$$

hence

$$\frac{|V(f)|}{|V(G)|} = \frac{|V(g)| + 2}{|V(H)| + 2} \geq \frac{|V(g)|}{|V(H)|} \geq \frac{4}{5},$$

or

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|.$$

*Subcase 1.2:*  $v_T \notin V(g)$ .

There are two edges  $e, e' \in \partial_H(v_T)$ , such that  $g(e) = g(e')$ . Let  $x = g(e)$ , and let  $y$  and  $z$  be two edges of  $S$  that are incident to the same end-vertex of  $x$  in  $S$ .

Consider a mapping  $f : E(G) \rightarrow S$ , obtained from  $g$  as follows: color the edges of  $T$  that are opposite to the edges with color  $x$  by  $y$ , and color the remaining third edge of  $T$  with  $z$ . Observe that

$$|V(f)| = |V(g)| + 2, \text{ and } |V(G)| = |V(H)| + 2,$$

hence

$$\frac{|V(f)|}{|V(G)|} = \frac{|V(g)| + 2}{|V(H)| + 2} \geq \frac{|V(g)|}{|V(H)|} \geq \frac{4}{5},$$

or

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|.$$

Moreover, for each vertex  $w \notin V(f)$ , there are two edges  $h, h' \in \partial_G(w)$ , such that  $f(h) = f(h')$ .

*Case 2:* All triangles of  $G$  are not contractible.

Let  $\mathcal{T}$  be the set of all triangles of  $G$ . Observe that  $\mathcal{T}$  can be empty. Consider a graph  $G'$  obtained from  $G$  by removing all vertices of  $G$  that lie on a triangle of  $\mathcal{T}$ . Observe that  $G'$  is a triangle-free graph of maximum degree at most 3.

We will use the labels of edges of  $S$  given in Figure 2. Consider a coloring of edges of  $G'$  with colors  $a, b$  and  $c$ , such that the number of uncolored edges is smallest. Let  $e$  be an uncolored edge. Color  $e$  with a color  $d$  from  $\{a, b, c\}$ , such that there is only one edge adjacent to  $e$ , such that it has also color  $d$ . Observe that all edges of  $G'$  are colored.

Now, we are going to extend this coloring to that of  $G$ . Choose a triangle  $T$  from  $\mathcal{T}$ . As  $T$  is not contractible, we have that the subgraph of  $G$  induced by the vertices of  $T$  form an end-block  $B$  of  $G$ . Moreover,  $B$  is isomorphic to end-blocks of  $S$ . Let  $v$  the root of  $B$ . Choose a color  $d \in \{a, b, c\}$  such that  $d$  is missing on the vertex  $v$  in the coloring of  $G'$ . Color the bridge joining a vertex of  $T$  to  $v$  by  $d$ , and color the edges of  $B$  by corresponding edges of the end-block of  $S$ , which contains a vertex incident to  $d$ . Let  $f$  be the resulting coloring.

Observe that all edges of  $G$  are colored in  $f$ . Moreover, vertices of  $V(G) \setminus V(f)$  lie in  $G'$ . Since  $G'$  is triangle-free, we have that the cycles corresponding uncolored edges are of length at least 5. Since they are vertex-disjoint (Lemma 3), we have that their number is at most  $\frac{|V(G')|}{5}$ . It is not hard to see that each uncolored edge  $e$  is incident to a vertex  $v$  such that  $v \in V(G) \setminus V(f)$ . Moreover,  $|V(G) \setminus V(f)|$  coincides with the number of uncolored edges, which implies that

$$|V(G) \setminus V(f)| \leq \frac{|V(G')|}{5} \leq \frac{|V(G)|}{5},$$

or

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|.$$

Finally, for each vertex  $w \notin V(f)$ , there are two edges  $h, h' \in \partial_G(w)$ , such that  $f(h) = f(h')$ .  
The proof of the theorem is completed.  $\square$

**Corollary 2.** *Let  $G$  be a cubic graph. Then, there is a mapping  $f : E(G) \rightarrow S$ , such that*

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|.$$

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